

Midterm COMP 2804

October 23, 2013

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.

Marking scheme: Each of the 17 questions is worth 1 mark.

- Newton: $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$.

1. Let A be a set of size 7 and let B be a set of size 13. How many one-to-one functions $f : A \rightarrow B$ are there?

- (a) $\frac{6!}{13!}$
- (b) $\frac{13!}{5!}$
- (c) $\frac{13!}{6!}$
- (d) $\frac{13!}{7!}$

2. You are given 5 books and 7 bookshelves. How many ways are there to place these books on the shelves? (The order on the shelves matters.)

- (a) $\binom{7}{5}$
- (b) $\frac{11!}{6!}$
- (c) $\frac{11!}{7!}$
- (d) $\frac{12!}{7!}$

3. A password consists of 6 or 7 characters, each character being an uppercase letter or a lowercase letter. A password must contain at least one uppercase letter. How many passwords are there?

- (a) $52^6 + 52^7$
- (b) $26 \cdot 52^5 + 26 \cdot 52^6$
- (c) $52^6 + 52^7 - 26^6 - 26^7$
- (d) none of the above

4. In a group of 20 people,

- 6 are blond,
- 7 have green eyes,
- 11 are not blond and do not have green eyes.

How many people are blond and have green eyes?

- (a) 3
- (b) 4
- (c) 5
- (d) 9

5. How many bitstrings of length 55 start with 101 or end with 1111?

- (a) $2^{52} + 2^{51}$
- (b) $2^{55} - 2^{48}$
- (c) $2^{55} - 2^{52} - 2^{51}$
- (d) $2^{52} + 2^{51} - 2^{48}$

6. Each person in a group of n people has a last name consisting of two uppercase letters. For what values of n can we guarantee that there are at least two people with the same last name?

- (a) $n \geq 26$
- (b) $n \geq 52$
- (c) $n \geq 676$
- (d) $n \geq 677$

7. How many bitstrings of length 13 contain exactly 3 zeros?

- (a) $\binom{13}{10}$
- (b) $13!/3!$
- (c) $2^{13} - \binom{13}{3}$
- (d) $2^{13} - 3$

8. What is the coefficient of $x^{12}y^{12}$ in the expansion of $(3x - 7y)^{24}$?

- (a) $-3^{12}7^{12}\binom{24}{12}$
- (b) $(3x)^{12}(-7y)^{12}\binom{24}{12}$
- (c) $21^{12}\binom{24}{12}$
- (d) $(3x)^{12}(7y)^{12}\binom{24}{12}$

9. Which of the following is true?

- (a) $\sum_{k=0}^n 5^k \binom{n}{k} = 6^n$
- (b) $\sum_{k=0}^n 4^{n-k} 5^k \binom{n}{k} = 8^n$
- (c) $\sum_{k=0}^n 5^k \binom{n}{k} = 5^n$
- (d) $\sum_{k=0}^n 4^k 5^{n-k} \binom{n}{k} = 20^n$

10. How many strings can be obtained by rearranging the letters of the word

POOPERSCOOPER

- (a) $13!$
- (b) $\binom{13}{4} \binom{9}{3} \binom{6}{2} \binom{4}{2} \binom{2}{1}$
- (c) $\binom{13}{4} \binom{9}{3} \binom{6}{2} \binom{4}{2}$
- (d) $4!3!2!2!1!1!$

11. The function $f : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 14 \\ f(n+1) &= f(n) + 4n - 5 \text{ for } n \geq 0 \end{aligned}$$

What is $f(n)$?

- (a) $f(n) = 2n^2 + 6n + 14$
- (b) $f(n) = 2n^2 - 6n + 14$
- (c) $f(n) = 2n^2 + 7n + 14$
- (d) $f(n) = 2n^2 - 7n + 14$

12. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

```
Algorithm FIB( $n$ ):  
  if  $n = 0$  or  $n = 1$   
  then  $f = n$   
  else  $f = \text{FIB}(n - 1) + \text{FIB}(n - 2)$   
  endif;  
  return  $f$ 
```

When running FIB(5), how many calls are there to FIB(2)?

- (a) 1
- (b) 2
- (c) 3
- (d) 4

13. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Consider again the recursive algorithm FIB, which takes as input an integer $n \geq 0$:

```
Algorithm FIB( $n$ ):  
  if  $n = 0$  or  $n = 1$   
    then  $f = n$   
    else  $f = \text{FIB}(n - 1) + \text{FIB}(n - 2)$   
    endif;  
  return  $f$ 
```

For $n \geq 2$, run algorithm FIB(n) and let a_n be the number of times that FIB(0) is called.

- (a) For $n \geq 2$, $a_n = f_{n-1}$
 - (b) For $n \geq 2$, $a_n = f_n$
 - (c) For $n \geq 2$, $a_n = f_n - 1$
 - (d) For $n \geq 2$, $a_n = f_{n+1}$
14. What does the summation $\sum_{k=7}^n \binom{k-1}{6}$ count?
- (a) The number of subsets of $\{1, 2, \dots, n\}$ having size 5.
 - (b) The number of subsets of $\{1, 2, \dots, n\}$ having size 6.
 - (c) The number of subsets of $\{1, 2, \dots, n\}$ having size 7.
 - (d) The number of pints of beer you drink when running algorithm BEER(n).
15. If you flip a fair coin 4 times, what is the probability that the coin comes up head exactly twice?
- (a) $1/\binom{4}{2}$
 - (b) $2/2^4$
 - (c) $2^4/\binom{4}{2}$
 - (d) $\binom{4}{2}/2^4$
16. If you choose an element x uniformly at random from the set $\{1, 2, \dots, 100\}$, what is the probability that x is divisible by 4 or 5?
- (a) 9/100
 - (b) 1/5
 - (c) 2/5
 - (d) 45/100

17. If you answer each question in this midterm by choosing an answer uniformly at random, what is the probability that you get all answers correct?

- (a) $1/17^4$
- (b) $1/4^{17}$
- (c) $3^{17}/4^{17}$
- (d) $4^{17}/3^{17}$

